NEGATIVE (LU AND UL) TAIL DEPENDENCE USING COPULAE

YURI SALAZAR FLORES

ABSTRACT. This paper studies the four types: lower, upper, lower-upper and upper-lower focusing on developping theory to measure the negative tail dependence

Introduction

As a consequence of globalization and relaxed regulations on financial and insurance markets the dependencies between financial time series have increased during recent times. Particularly in extreme events such as economic crisis, these dependencies affect the profit of companies as well as the financial stability of the financial sector. Risk measures such as VaR (Value at Risk) also depend on the dependence structure of extreme values. In order to measure these dependencies, tail dependence measures are used, these measures rely on copulae, tail copulae and the tail dependence coefficient (TDC) in order to explain the dependence structure of extreme values (see Embrechts et al. (1997)).

In the study of financial time series it has become increasingly important to distinguish between different types of dependence. The structure dependence of time series have been studied for a long time, traditionally through the use of correlation. Due to drawbacks of this measure new methodologies have been developped, in particular the use of copulae has proved to be the way forward (see McNeil et al. (2005), Chapter 5), however this analysis mainly focuses on positive correlation by the use of usual copulae and survival copulae. The use of copulae has been particularly succesful to measure dependence on the extreme values in what is known as tail dependence through the use of tail copulae and the tail dependence coefficients (TDCs), but yet again the analysis has been mostly directed to the left (lower tail dependence) and the right tail (upper tail dependence). Many times, when analysing financial time series it happens that two series present the same type of extreme behaviour, either upper or lower tail dependence. However it can also happen that a value in the left tail in one series may occur at the same time when a value in the right tail of the other series appears. This can be observed in prices of stocks that affect a certain portfolio as well as in stock indices returns and other financial time series. We will refer to this as lower-upper (LU) tail dependence or upper-lower (UL) tail dependence. Although UL tail dependence may seem to be covered by LU tail dependence, in most of the cases, it is worth to be studied separately. The use of estimators for the UL and LU TDCs has already observed in finance (see Zhang (2007)), In this paper we define new copulae associated to the sample that enable us to capture the whole dependence structure of the series.

This work is divided in two sections:In the first section the theory of tail dependence is studied and theory on LU and UL tail dependence is developed from probability functions, copulae, tail copulae and the TDC, mathematical proofs are provided on main results. In order to study LU and UL tail dependence it is necessary to work with certain probability functions, which we call LU and UL probability functions.Using this probability functions we introduce new types of copulae, LU and UL copulae. The first results are related to the equalities connecting these copulae and then to connect it to usual and survival copulae. Given that the usual copula is the copula of distribution functions, to differentiate it from survival and other copulae, we will refer to it as distributional copula. The boundaries of copulae are used to restrict LU and UL probability functions and results on exchangeable copulae are presented.

In the second section upper, lower, LU and UL tail dependence are modelled, we revise some of the most important copulae models discussed in literature. The first examples of copulae we study are the fundamental copulae which encompasses three cases: the independent case with the independent copula, the perfect positive dependence case with the comonotonicity copula and the perfect negative dependence case with the countercomonotonicity copula. We then study two examples of implicit copulae, for which there is no closed form, the Gaussian and the Student's t copula. After that we study the archimidean copulae, such as the Gumbel, Clayton, Frank and the Generalized Clayton copula. Finally we also study a non-archimidean copula which is the Marshal-Olkin copula. For all examples we present their corresponding tail copulae and TDCs to see if they account for tail dependence.

1. Copulae, Tail Copulae and the Tail Dependence Coefficients

1.1. Copula.

In order to define and study the dependence structure between two random variables we use the concept of copula. The following study is based on copulae which describe the dependence structure of multidimensional random variables. Here we restrict to the continuous two dimensional case. We first study the theory of copula and introduce the LU and UL copulae along with results regarding these copulae and then we focus on their relationship with distributional and survival copulae. After this we study Tail Copulae and the TDC for the four types of copula: distributional, survival, LU and UL.

The concept of copula was first introduced by Sklar (1959), and is now a cornerstone topic in finance (see Nelsen (1999) or McNeil et al. (2005), Chapter 5), a two dimensional copula is defined in the following way:

Definition 1. *A two dimensional copula $C(u, v)$ is a distribution function on $[0, 1]^2$ with standard uniform marginal distributions.*

In the two dimensional case copulae functions $C : [0, 1]^2 \to [0, 1]$ are used to link bivariate distribution functions with their corresponding marginal distributions. On the other hand, survival copulae: $\overline{C} : [0, 1]^2 \to [0, 1]$ link bivariate survival functions with their corresponding marginal survival functions.

Let $(X;Y)'$ be a random vector with joint distribution function $F(x,y) = P(X \leq x, Y \leq y)$, marginals $G(x) = P(X \leq x)$, $H(y) = P(Y \leq y)$, survival function $\overline{F}(x,y) = P(X > x, Y > y)$ and marginal survival functions , $\overline{G}(x) = P(X > x)$ and $\overline{H}(y) = P(Y > y)$. Two versions of Sklar's theorem guarantee the existence and uniqueness of copulae C and $\overline{C}$ (see Schweizer and Sklar (1983))

$$F(x,y) = C(G(x), H(y)), \text{ which is equivalent to } C(u,v) = F(G^{-1}(u), H^{-1}(v)) \tag{1.1}$$

and

$$\overline{F}(x,y) = \overline{C}(\overline{G}(x), \overline{H}(y)), \text{ which is equivalent to } \overline{C}(u,v) = \overline{F}(\overline{G}^{-1}(u), \overline{H}^{-1}(v)). \tag{1.2}$$

Given that $\overline{G}^{-1}(u) = G^{-1}(1-u)$, equation (1.2) is also equivalent to $\overline{C}(u,v) = F(G^{-1}(1-u), H^{-1}(1-v))$.

1.1.1. ***Sklar's Theorem for LU and UL Probability Functions.*** In this case we refer to C as distributional copula and to $\overline{C}$ as survival copula. In the same way that a distributional copula and a survival copula explain the dependence structure of two random variables between their left and right tails respectively, we now introduce a new type of copula to explain the dependence structure of two random variables between the left tail of the first one and the right tail of the other one, and vice versa.

Definition 2. *Let $(X,Y)'$ be a random vector, its lower-upper (LU) and upper-lower (UL) probability funtions are $F_{LU}(x,y) = P(X \leq x, Y > y)$ and $F_{UL}(x,y) = P(X > x, Y \leq y)$. If G, H, $\overline{G}$ and $\overline{H}$ are the disbtributions and survival functions of X and Y we refer to G and $\overline{H}$ as the marginals of F_{LU} and to $\overline{G}$ and H as the marginals of F_{UL}*

The copulas we consider are C_{LU} and C_{UL} that link the functions in definition (2) with their corresponding marginals. We refer to them as copulae of the LU and UL probability functions or simply LU and UL copulae. The following version of Sklar's theorem guarantees the existence and uniqueness of C_{LU} and C_{UL} in the continuous case. A more general version can be stated following the same reasoning of the proof of Skar's theorem in the non-continuous case, but here we restrict to this case (see Schweizer and Sklar (1983) or Nelsen (1999), p. 18).

Theorem 1. *Sklar's theorem for lower-upper and upper-lower probability funtions.*

Let $(X,Y)'$ be a random vector, F_{LU} and F_{UL} its LU and UL probability functions as in definition (2) and the distribution functions of X and Y, G and H be continuous, then there exist unique copulae C_{LU} and $C_{UL} : [0,1]^2 \to [0,1]$, such that, for all x and y in $[-\infty, \infty]$,

$$F_{LU}(x,y) = C_{LU}(G(x), \overline{H}(y)), \tag{1.3}$$

$$F_{UL}(x,y) = C_{UL}(\overline{G}(x), H(y)). \tag{1.4}$$

Conversely, if we have any copulae C_{LU} and C_{UL} satisfying definition (1) and G, H, $\overline{G}$ and $\overline{H}$ univariate disbtributions and its survival functions then, considering the previous equations, F_{LU} defines a LU probability function with marginals G and $\overline{H}$ and F_{UL} defines an UL probability function with marginals $\overline{G}$ and H.

Proof. The proof of this theorem is analogous to the proof of Sklar's theorem in the continuous case, (see McNeil et al (2005), p.186). From Defintion (2) and considering that $P(X \le x) = P(F(X) \le F(x))$ for any distribution function F, we have that for any x and y in $[-\infty, \infty]$

$$F_{LU}(x,y) = P(G(X) \le G(x), \overline{H}(Y) \le \overline{H}(y)).$$

Using the continuity of G and $\overline{H}$ (see McNeil et al. (2005), proposition (5.2 (2)), p.185), $G(X)$ and $H(Y)$ are uniformly distributed, which implies $1 - H(Y)$ is uniformly distributed, so we have that both $G(X)$ and $\overline{H}(Y)$ have standard uniform distributions. Definition (1) implies that the distribution function of $(G(x), \overline{H}(y))$ is a copula. We denote this copula by C_{LU}, yielding equation (1.3). Evaluating this equation in the generalized inverses $G^{\leftarrow}(u)$ and $H^{\leftarrow}(1-v)$ for $u, v \in [0,1]$ and using the fact that one of the properties of the generalized inverse is that when T is continuous $T \circ T^{\leftarrow}(x) = x$, we get:

$$C_{LU}(u,v) = F_{LU}(G^{\leftarrow}(u), H^{\leftarrow}(1-v)) \quad ,$$

which explicitly represents C_{LU} in terms of F_{LU} and its marginals implying its uniqueness.

For the converse statement of the theorem let C_{LU} be any copula satisfying definition (1), $W = (U, V)$ a random vector with distribution function C_{LU} and G and $\overline{H}$ univariate distribution and survival functions. We now define $Z = (X, Y) := (G^{\leftarrow}(U), H^{\leftarrow}(1-V))$. Considering that another property of the generalized inverse is that if T is right continuous, like distribution functions, $T(x) \geq y \iff T^{\leftarrow}(y) \leq x$, the LU probabilty function of Z is

$$P(X \leq x, Y > y) = C_{LU}(G(x), \overline{H}(y)).$$

This implies that F_{LU} defined in (1.3) is the LU probability function of Z with marginals $P(X \leq x) = P(G^{\leftarrow}(U) \leq x) = P(U \leq G(x)) = G(x)$ and $P(Y > y) = P(H^{\leftarrow}(1-V) > y) = P(V \leq \overline{H}(y)) = \overline{H}(y)$. Note that for this theorem we refered to generalized inverses rather than inverse functions, as the first are more general. However throughout this work, whenever we are not proving a general property, we assume the distribution functions have inverse functions. For the properties of the generalized inverse function used in this proof (see McNeil et al (2005), proposition (A.3)) □

Note that this theorem implies that in the continuous case C_{LU} and C_{UL} are the LU and UL probability functions of $(G(X), \overline{H}(Y))$ and $(\overline{G}(x), H(y))$ characterized as:

$$F_{LU}(x,y) = C_{LU}(G(x), \overline{H}(y)),\ C_{LU}(u,v) = F_{LU}(G^{-1}(u), H^{-1}(1-v)) \tag{1.5}$$

and

$$F_{UL}(x,y) = C_{UL}(\overline{G}(x), H(y)),\ C_{UL}(u,v) = F_{UL}(G^{-1}(1-u), H^{-1}(v)). \tag{1.6}$$

1.1.2. ***Properties of Distributional, Survival, LU and UL Copulae.*** As we know from definition (1) the copulae we have studied so far are distribution functions with uniform marginals, the following proposition characterizes their corresponding random vectors.

Proposition 1. *Let (X, Y) be a random vector with continuous marginal distribution functions G and H, with corresponding distributional, survival, LU and UL copulae C, $\overline{C}$, C_{LU} and C_{UL} and let $(U, V) = (G(X), H(Y))$ then C is the distribution function of (U, V), $\overline{C}$ of $(1-U, 1-V)$, C_{LU} of $(U, 1-V)$ and C_{UL} of $(1-U, V)$*

Proof. Using equations (1.1), (1.2), (1.5) and (1.6) we evaluated the corresponding distribution functions in terms of the copulae and obtained

$$\begin{aligned} P(U \le u, V \le v) &= C(u,v), \\ P(1-U \le u, 1-V \le v) &= \overline{C}(u,v), \\ P(U \le u, 1-V \le v) &= C_{LU}(u,v), \\ P(1-U \le u, V \le v) &= C_{UL}(u,v). \end{aligned}$$

□

Similar to a distributional copula (see McNeil et al (2005), proposition (5.6)), in the continuous case the survival, the LU and UL copulae are also invariant under strictly increasing transformations. We state this in this proposition:

Proposition 2. *Let $T_!$ and T_2 be strictly increasing functions and (X, Y) a random vector with corresponding distributional, survival, LU and UL copulae C, $\overline{C}$, C_{LU} and C_{UL} then $(T_1(X), T_2(Y))$ also has the same corresponding copulae.*

Proof. We prove this for the survival, the LU and UL copulae using the same arguments used for the distributional copula. Let $\widetilde{G}(u) := G \circ T_1^{\leftarrow}(u)$ and $\widetilde{H}(v) := H \circ T_2^{\leftarrow}(v)$.

i) $\widetilde{G}$ and $\widetilde{H}$ are the respective distribution functions of $T_1(X)$ and $T_2(Y)$.

ii) $T_i^{\leftarrow} \circ T_i(x) = x$, $i = 1, 2$, which implies $G(u) = \widetilde{G}(T_1(u))$ and $H(v) = \widetilde{H}(T_2(v))$.

iii) $D \circ D^{\leftarrow}(x) = x$ for any univariate continuous distribution function D.

(See McNeil et al. (2005) propositions (5.6), and (A.3 vii) and (viii)). Given that (i) holds, as a notation, we use the tilde $\tilde{}$ to denote the probability functions associated to $(T_1(X), T_2(Y))$. Considering the properties mentioned above and using equations (1.1), (1.2), (1.5) and (1.6) that define copulae, we have:

$$\begin{aligned} \overline{C}(u, v) &= \overline{F}(\overline{G}^{\leftarrow}(u), \overline{H}^{\leftarrow}(v)) \\ &= \widetilde{\overline{F}}(\widetilde{\overline{G}}^{\leftarrow}(u), \widetilde{\overline{H}}^{\leftarrow}(v)) \\ &= \widetilde{\overline{C}}(u, v). \end{aligned}$$

Also

$$\begin{aligned} C_{LU}(u, v) &= F_{LU}(G^{\leftarrow}(u), \overline{H}^{\leftarrow}(v)) \\ &= \widetilde{F_{LU}}(\widetilde{G}^{\leftarrow}(u), \widetilde{\overline{H}}^{\leftarrow}(v)) \\ &= \widetilde{C_{LU}}(u, v). \end{aligned}$$

And:

$$\begin{aligned} C_{UL}(u,v) &= F_{UL}(\overline{G}^{\leftarrow}(u), H^{\leftarrow}(v)) \\ &= \widetilde{F_{UL}}(\widetilde{\overline{G}}^{\leftarrow}(u), \widetilde{H}^{\leftarrow}(v)) \\ &= \widetilde{C_{UL}}(u,v). \end{aligned}$$

For the expressions of $\overline{C}$, C_{LU} and C_{UL} we used: (iii), (ii), (i), that survival functions and their generalized inverses are non-increasing, the continuity of the random variables and equations (1.2), (1.5) and (1.6). These results prove that the copulae associated to (X,Y) are the same as the copulae associated to $(T_1(X), T_2(Y))$. Note that given that this is a general statement we used $^{\leftarrow}$ instead of $^{-1}$ as this statement holds for the general inverse function. □

Just as other copulae the LU and UL copulae satisfy the Fréchet bounds for copulae

$$\max\{u+v-1,0\} \leq C_{LU}(u,v), C_{UL}(u,v) \leq \min\{u,v\}.$$

These bounds allow us to establish bounds for the LU and UL probability functions in terms of its marginals and hence in terms of the associated distribution functions, i.e.

$$\begin{aligned} G(x) - \min\{G(x), H(y)\} &\leq F_{LU}(x,y) \leq \min\{G(x), 1-H(y)\}, \quad (1.7) \\ H(y) - \min\{G(x), H(y)\} &\leq F_{UL}(x,y) \leq \min\{1-G(x), H(y)\}. \end{aligned}$$

A well known relationship that links distributional copulae and survival copulae is

$$\overline{C}(u,v) = u+v-1+C(1-u,1-v) \quad (1.8)$$

The same relationship holds for LU and UL copulae, furthermore it is possible to link LU and UL copulae with distributional and survival copulae, we show and prove such relationships in the next proposition.

Proposition 3. *Let (X, Y) be a random vector with continuous marginal distribution functions G and H, C its associated distributional copula, $\overline{C}$ its survival copula and C_{LU} and C_{UL} its associated LU and UL copulae, then the following relationships hold*

$$C_{LU}(u,v) = u + v - 1 + C_{UL}(1-u, 1-v), \tag{1.9}$$

$$u = C_{LU}(u,v) + C(u, 1-v) = C_{UL}(u,v) + \overline{C}(u, 1-v), \tag{1.10}$$

$$v = C_{LU}(u,v) + \overline{C}(1-u, v) = C_{UL}(u,v) + C(1-u, v). \tag{1.11}$$

Proof. The previous equations follow from the fact that in the continuous case both $G(X)$ and $H(Y)$ follow standard uniform distributions. Let us consider the events

$$A_1 = \{X \leq G^{\leftarrow}(u)\},\ B_1 = \{Y \leq H^{\leftarrow}(v)\},$$

$$A = \{X > G^{\leftarrow}(1-u)\},\ B = \{Y > H^{\leftarrow}(1-v)\}.$$

The probabilities of this events are obtained as, $P(A) = P(A_1) = u$ and $P(B) = P(B_1) = v$. To prove the proposition we use the representation of copulae in equations (1.1), (1.2), (1.5), (1.6), and set theory. For (1.9) we have that

$$\begin{aligned} C_{LU}(u,v) &= P(A_1 \cap B) \\ &= 1 - [P(A_1^c) + P(B^c) - P(A_1^c \cap B^c)] \\ &= u + v - 1 + C_{UL}(1-u, 1-v). \end{aligned}$$

The other equations are proved in a similar way

$$\begin{aligned} u &= P(A_1 \cap B) + P(A_1 \cap B^c) \\ &= C_{LU}(u,v) + C(u, 1-v), \end{aligned}$$

$$\begin{aligned} u &= P(A \cap B_1) + P(A \cap B_1^c) \\ &= C_{UL}(u,v) + \overline{C}(u, 1-v), \end{aligned}$$

$$\begin{aligned} v &= P(A_1 \cap B) + P(A_1^c \cap B) \\ &= C_{LU}(u,v) + \overline{C}(1-u,v), \end{aligned}$$

$$\begin{aligned} v &= P(A \cap B_1) + P(A^c \cap B_1) \\ &= C_{UL}(u,v) + C(1-u,v). \end{aligned}$$

□

Note that an equivalent equation to (1.9) for C_{UL} is also valid if we substitute u and v by $(1-u)$ and $(1-v)$ in this equation. Equations (1.10) and (1.11) are meant to express C_{LU} and C_{UL} in terms of other copulae. If we want to do the opposite, we would have to substitute v by $(1-v)$ in equation (1.10) and u by $(1-u)$ in equation (1.11). This proposition is very helpful when we study LU and UL tail dependence in the case when we know the distributional copula.

1.1.3. ***Transverse Copulae.*** Proposition (3) states specific relationships among all four copulae we have defined so far. Equation (1.9) proves that the relationship C_{LU}-C_{UL} is the same as the relationship C-$\overline{C}$. Equation (1.10) proves the same for the relationships C_{LU}-C and C_{UL}-$\overline{C}$ and (1.11) for C_{LU}-$\overline{C}$ and C_{UL}-C. In order to characterize such relationships we define:

Definition 3. *If C is a copula according to definition (1), its transverse copula C^T is defined as*

$$C^T(u,v) := u + v - 1 + C(1-u, 1-v).$$

Its U-transverse copula is C^U defined as

$$C^U(u,v) := v - C(1-u,v).$$

And its V-transverse copula C^V is

$$C^V(u,v) := u - C(u,1-v).$$

Note that the three types of transversity are reflexible, meaning that if one copula C is the transverse copula of C^*, in any of the three kinds of transversity, then C^* is the transverse copula of C, in the same type of transversity. To check this, replace u and v with $(1-u)$ and $(1-v)$ in the first equality, u with $(1-u)$ in the second one and v with $(1-v)$ in the third one. We say two copulae are transverse.

Propositions (1) and (3) not only imply that transverse copulae are well defined in the sense that they are indeed copulae according to definition (1), they also provide a specific expression of the random vectors of which they are distribution functions in terms of (U,V), the random vector with distribution function C. We prove this in the following proposition.

Proposition 4. *Let C be a copula according to definition (1), and let (U,V) be the random vector with distribution function C, then the transverse copulae of definition (3) are all copulae according to definition (1) and they satisfy that C^T is the distribution function of $(1-U,1-V)$, C^U of $(1-U,V)$ and C^V of $(U,1-V)$.*

Proof. $(X,Y) := (U,V)$ is a random vector with marginals $G = H = I$ and corresponding distributional copula C. Sklar's theorem guarantees the existence and uniqueness of the corresponding survival, LU and UL copulae, $\overline{C}$, C_{LU} and C_{UL}. Using their uniqueness, equation (1.8) and proposition (3) we know that: $\overline{C} = C^T$, $C_{LU} = C^V$ and $C_{UL} = C^U$ which implies that they are copulae. Definition (1) finishes the proof. □

It follows from proposition (4) that C^U and C^V are transverse, C^U and C^T are V-transverse and C^V and C^T are U-transverse, this can also be checked using definition (3). Proposition (3) provides the transversity relationships for the copulae associated to a random vector, the concept of transversity is used to characterize the relationships among the copulae associated to a random vector. To illustrate this, in Table 1 we present two cartesian planes with a horizontal axis, that we

V			V		
C^V	C^T	U	C_{LU}	$\overline{C}$	U
C	C^U		C	C_{UL}	

TABLE 1. Relationship Structure of Copulae

call the U axis and a vertical axis, that we call the V axis, each one of the four quadrants has a positive or a negative sign on each axis.

Table 1 characterizes the relationship structure of the copulae associated to a random vector. Two copulae are transverse when they are in a diagonal, U-transverse when they have different sign on the U axis and V transverse when they have different sign on the V axis. The first part of the table comes from proposition (4) and the second one from definition (3) and proposition (3).

Table is very useful when we first set a specific copula C^* as distributional copula and then evaluate the corresponding $\overline{C^*}$, C^*_{LU}, and C^*_{UL}. If we want to set C^* as say LU copula we do not have obtain the other copulae again because we already know the relationship structure of all four copulae from Table 1. Using the concept of transversity we can explain the whole relationship structure among the distributional, survival, LU and UL copulae. We obtain results on transversity such as proposition (5) that we later apply to the copulae associated to a random vector. Another useful concept that we use later is exchangeability which we now define in the bivariate case:

Definition 4. *A random vector (X, Y) is said to be exchangeable if $(X, Y) \stackrel{d}{=} (Y, X)$. A copula C is said to be exchangeable if it is the distribution function of an exchangeable vector, in this case the copula satisfies $C(u, v) = C(v, u)$*

The following proposition provides equivalences for the exchangeability of transverse copulae and follows from definition (3):

Proposition 5. *Let C be a copula with corresponding transverse copulae as in definition (3), then the following equivalences hold:*

i) C is exchangeable $\iff$ C^T is exchangeable $\iff$ $C^U(u,v) = C^V(v,u)$.

ii) $C \equiv C^T \iff C^U \equiv C^V$, $C \equiv C^U \iff C^V \equiv C^T$ and $C \equiv C^V \iff C^U \equiv C^T$.

iii) C and C^V are exchangeable $\implies$ $C \equiv C^T$ and $C^U \equiv C^V$.

Proof. i) We prove this by proving $A \iff B$ and $A \iff C$

$A \iff B$ follows from the definition of transverse copula from which we can express both copulae in terms of each other.

For $A \iff C$ we know from definition (3) that

$$C^U(u,v) - C^V(v,u) = C(v, 1-u) - C(1-u, v). \tag{1.12}$$

Equation (1.12) proves the equivalence, because if C is exchangeable then the right hand side is zero implying the desired equation. Now if $C^U(u,v) = C^V(v,u)$ then the left hand side is zero proving that C is exchangeable.

ii) From definition (3) we know that

$$\begin{aligned} C^U(u,v) - C^V(u,v) &= C^T(1-u, v) - C(1-u, v), \\ C^V(u,v) - C^T(u,v) &= C^U(u, 1-v) - C(u, 1-v), \\ C^U(u,v) - C^T(u,v) &= C^V(1-u, v) - C(1-u, v). \end{aligned} \tag{1.13}$$

Equation (1.13) implies the equivalences using the same arguments we used in (1.12) for (i).

iii) Using the assumption and equation (1.12) we know that

$$C^U(u,v) - C^V(u,v) = C(v, 1-u) - C(v, 1-u) = 0.$$

Hence, $C^U \equiv C^V$ and (ii) implies $C \equiv C^T$. □

In general copulae are used to explain dependence structure between tails of random variables, LU and UL copulae are useful to explain the dependence structure between the left tail of one random variable and the right tail of the other one, however when it comes to measure this dependence "deep" into the tail, the concept of tail copula is oftenly used, (see Embrechts et al. (1997) or Schmidt and Stadtmüller (2006)). It must be said that tail copulae are not copula according to definition (1) but there are a number of reasons that justify this name, including the fact that is defined as a limit of copulae and that it describes the dependence structure of the tails in a similar way to the way a copula does.

1.2. **Tail Copulae and the Tail Dependence Coefficients.** Both tail copulae and the TDCs describe the dependence structure between two random variables in the tail. Equation (1.14) shows how tail copulae captures such structure. Tail copulae was introduced as a generalization of the case $u = v = 1$, that corresponds to the TDC (see Schmidt and Stadtmüller (2006)).

In equation (1.20) we see, in an intuitive way, why this coefficient is used to measure tail dependence. Because of this equation (1.14) and (1.20) can be used as definitions of tail copula and TDC. We use definitions (11) and (12) as they are easier to deal with when it comes to obtaining mathematical results, however it must be noted that they are equivalent.

Definition 5. *Let $Z = (X, Y)'$ be a random vector with corresponding copulae C, $\overline{C}$, C_{LU} and C_{UL}. Whenever the following limits exist on $\overline{\mathbb{R}}^2_+ := [o, \infty]^2 \backslash \{(\infty, \infty)\}$*

$$\begin{aligned}
\Lambda_L(u, v) \quad &: \quad = \lim_{t \to \infty} tC\left(\frac{u}{t}, \frac{v}{t}\right), \\
\Lambda_U(u, v) \quad &: \quad = \lim_{t \to \infty} t\overline{C}\left(\frac{u}{t}, \frac{v}{t}\right), \\
\Lambda_{LU}(u, v) \quad &: \quad = \lim_{t \to \infty} tC_{LU}(\left(\frac{u}{t}, \frac{v}{t}\right)), \\
\Lambda_{UL}(u, v) \quad &: \quad = \lim_{t \to \infty} tC_{UL}\left(\frac{u}{t}, \frac{v}{t}\right).
\end{aligned}$$

The functions Λ_L, Λ_U, Λ_{LU}, $\Lambda_{UL} : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ *are called the lower, upper, lower-upper and upper-lower tail copulae associated to* Z*. In general, if* $\Lambda(u,v) := \lim_{t\to\infty} tC(\frac{u}{t}, \frac{v}{t})$*, with the same assumptions as before and with* C *any copula according to definition (1) we call* $\Lambda : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ *a tail copula. We say that two tail copulae are transverse, any of the kinds of transversity in definition (3), if their underlying copulae are transverse.*

Note that from this definition it is clear that proposition (5) holds for tail copulae.

Let $U = G(X)$ and $V = H(Y)$. The following relationships regarding tail copulae and conditional probabilities are useful when defining estimatiors for tail copulae (see Schmidt and Stadtmüller (2006), p. 309):

$$\begin{aligned}
\Lambda_L(u,v) &= v \lim_{t\to\infty} P\left(U \le \frac{u}{t} | V \le \frac{v}{t}\right) = u \lim_{t\to\infty} P\left(V \le \frac{v}{t} | U \le \frac{u}{t}\right), \qquad (1.14)\\
\Lambda_U(u,v) &= v \lim_{t\to\infty} P\left(U \ge 1-\frac{u}{t} | V \ge 1-\frac{v}{t}\right) = u \lim_{t\to\infty} P\left(V \ge 1-\frac{v}{t} | U \ge 1-\frac{u}{t}\right),\\
\Lambda_{LU}(u,v) &= v \lim_{t\to\infty} P\left(U \le \frac{u}{t} | V \ge 1-\frac{v}{t}\right) = u \lim_{t\to\infty} P\left(V \ge 1-\frac{v}{t} | U \le \frac{u}{t}\right),\\
\Lambda_{UL}(u,v) &= v \lim_{t\to\infty} P\left(U \ge 1-\frac{u}{t} | V \le \frac{v}{t}\right) = u \lim_{t\to\infty} P\left(V \le \frac{v}{t} | U \ge 1-\frac{u}{t}\right).
\end{aligned}$$

It is sometimes useful to express the tail copula as $\Lambda_L(u,v) = \lim_{h\to 0} \frac{C(hu,hv)}{h}$. We use the function $Q : h \to C(hu, hv)$, which is a function that goes from $\mathbb{R} \to \mathbb{R}$, to obtain useful expressions for tail copulae. The chain rule states that whenever the partial derivatives of C exist, the derivative of Q with respect to h can be obtained in terms of C's partial derivatives:

$$\frac{dQ}{dh} = \frac{\partial C}{\partial u}(hu, hv) \cdot u + \frac{\partial C}{\partial v}(hu, hv) \cdot v.$$

Then using l'Hoppital's theorem we know that $\Lambda_L(u,v) = \lim_{h\to 0} \frac{dQ}{dh}$. Furthermore, again ussing the chain rule, and equations (1.8), (1.10) and (1.11), we can express the other tail copulae as well in terms of the partial derivatives of C

$$\Lambda_L(u,v) = u\cdot\left(\lim_{h\to 0}\frac{\partial C}{\partial u}(hu,hv)\right) + v\cdot\left(\lim_{h\to 0}\frac{\partial C}{\partial v}(hu,hv)\right), \tag{1.15}$$
$$\Lambda_U(u,v) = u+v-u\cdot\left(\lim_{h\to 0}\frac{\partial C}{\partial u}(1-hu,1-hv)\right) - v\cdot\left(\lim_{h\to 0}\frac{\partial C}{\partial v}(1-hu,1-hv)\right),$$
$$\Lambda_{LU}(u,v) = u-u\cdot\left(\lim_{h\to 0}\frac{\partial C}{\partial u}(hu,1-hv)\right) + v\cdot\left(\lim_{h\to 0}\frac{\partial C}{\partial v}(hu,1-hv)\right),$$
$$\Lambda_{UL}(u,v) = v+u\cdot\left(\lim_{h\to 0}\frac{\partial C}{\partial u}(1-hu,hv)\right) - v\cdot\left(\lim_{h\to 0}\frac{\partial C}{\partial v}(1-hu,hv)\right).$$

This expresion is particularly helpful when C has a closed form. It also enables us to express a tail copula in terms of conditional probabilities using the following relationship (see McNeil et al. (2005), eq 5.15, p. 196):

$$\begin{aligned} C_{V|U}(v|u) &= P(V\le v|U=u) \\ &= \frac{\partial C}{\partial u}(u,v) \end{aligned}$$

And also $C_{U|V}(u|v) = \frac{\partial C}{\partial v}(u,v)$. In the case when $U=G(X)$ and $V=H(Y)$ we have

$$\begin{aligned} \frac{\partial C}{\partial u}(u,v) &= P_{Y|X}(H^{-1}(v)|G^{-1}(u)) \\ \frac{\partial C}{\partial v}(u,v) &= P_{X|Y}(G^{-1}(hu)|H^{-1}(hv)). \end{aligned} \tag{1.16}$$

In this case, the expresion for Λ_L is:

$$\Lambda_L(u,v) = u\cdot(\lim_{h\to 0}P_{Y|X}(H^{-1}(hv)|G^{-1}(hu))) + v\cdot(\lim_{h\to 0}P_{X|Y}(G^{-1}(hu)|H^{-1}(hv))). \tag{1.17}$$

(analogous expressions for the other tail copulae can be obtained in the same way from equations (1.16) and (1.15)). If the copula C is exchangeable, it holds that $\frac{\partial C}{\partial v}(u,v) = \frac{\partial C}{\partial u}(v,u)$ and proposition (5i) implies that $C_{LU}(u,v) = C_{UL}(v,u)$. Thus, we can get equation (1.15) only in terms of $\frac{\partial C}{\partial u}$

$$\Lambda_L(u,v) = u \cdot \left(\lim_{h\to 0} \frac{\partial C}{\partial u}(hu,hv) \right) + v \cdot \left(\lim_{h\to 0} \frac{\partial C}{\partial u}(hv,hu) \right), \tag{1.18}$$
$$\Lambda_U(u,v) = u + v - u \cdot \left(\lim_{h\to 0} \frac{\partial C}{\partial u}(1-hu,1-hv) \right) - v \cdot \left(\lim_{h\to 0} \frac{\partial C}{\partial u}(1-hv,1-hu) \right),$$
$$\Lambda_{LU}(u,v) = u - u \cdot \left(\lim_{h\to 0} \frac{\partial C}{\partial u}(hu,1-hv) \right) + v \cdot \left(\lim_{h\to 0} \frac{\partial C}{\partial u}(1-hv,hu) \right),$$
$$\Lambda_{UL}(u,v) = \Lambda_{LU}(v,u).$$

In the exchangeable case, we have the following expression for the lower tail copula

$$\Lambda_L(u,v) = u \cdot (\lim_{h\to 0} P_{Y|X}(H^{-1}(hv)|G^{-1}(hu))) + v \cdot (\lim_{h\to 0} P_{Y|X}(G^{-1}(hu)|H^{-1}(hv))) \tag{1.19}$$

(other tail copulae can be expressed similarly using equations(1.18) and (1.16)). This expression is particularly useful to determine the other three tail copulae when we have a Gaussian or a Student's t copula.

Determining whether or not tail dependence is present and measuring dependence between extreme values have been done for sometime using the concept of tail dependence (see Schmidt and Stadtmüller (2006)). This has been done by introducing a tail dependence coefficient (TDC) which has the advantage of being very intuitive and concise, we now introduce such concepts for all copulae we have been studying.

Definition 6. *Let $Z = (X,Y)'$ be a random vector with corresponding copulae C, $\overline{C}$, C_{LU} and C_{UL}. Its corresponding lower, upper, lower-upper and upper-lower tail* *dependent coefficients (TDCs), is defined, whenever such limits exist, as*

$$\begin{aligned}
\lambda_L &:= \lim_{t\to\infty} tC\left(\frac{1}{t},\frac{1}{t}\right),\\
\lambda_U &:= \lim_{t\to\infty} t\overline{C}\left(\frac{1}{t},\frac{1}{t}\right),\\
\lambda_{LU} &:= \lim_{t\to\infty} tC_{LU}\left(\frac{1}{t},\frac{1}{t}\right),\\
\lambda_{UL} &:= \lim_{t\to\infty} tC_{UL}\left(\frac{1}{t},\frac{1}{t}\right).
\end{aligned}$$

Z is said to be lower, upper, lower-upper or upper-lower tail dependent when its corresponding $\lambda > 0$ and independent when its corresponding $\lambda = 0$. In general, if $\lambda := \lim_{t\to\infty} tC(\frac{1}{t},\frac{1}{t})$, with the same assumptions as before and with C any copula according to definition (1), we call λ a tail dependence coefficient. We say that two TDC are transverse, any of the kinds of transversity in definition (3), if their underlying copulae are transverse.

Note that proposition (5) can be used to determine some cases in which the TDCs are equal. If the corresponding tail copula Λ exists, the TDC is $\lambda = \Lambda(1,1)$. We use this to present the following version of equation (1.14), which shows a more intuitive way of how the TDC captures the tail dependence structure between two random variables

$$\begin{aligned}
\lambda_L &= \lim_{t\to\infty} P\left(U \le \frac{1}{t} | V \le \frac{1}{t}\right) = \lim_{t\to\infty} P\left(V \le \frac{1}{t} | U \le \frac{1}{t}\right), \qquad (1.20)\\
\lambda_U &= \lim_{t\to\infty} P\left(U \ge 1-\frac{1}{t} | V \ge 1-\frac{1}{t}\right) = \lim_{t\to\infty} P\left(V \ge 1-\frac{1}{t} | U \ge 1-\frac{1}{t}\right),\\
\lambda_{LU} &= \lim_{t\to\infty} P\left(U \le \frac{1}{t} | V \ge 1-\frac{1}{t}\right) = \lim_{t\to\infty} P\left(V \ge 1-\frac{1}{t} | U \le \frac{1}{t}\right),\\
\lambda_{UL} &= \lim_{t\to\infty} P\left(U \ge 1-\frac{1}{t} | V \le \frac{1}{t}\right) = \lim_{t\to\infty} P\left(V \le \frac{1}{t} | U \ge 1-\frac{1}{t}\right).
\end{aligned}$$

In the case $U = G(X)$ and $V = H(Y)$, we can use the inverse of the distribution functions, the quantile functions, to express the coefficients in terms of (X, Y) and the quantile functions.

As we mentioned before, tail copula was introduced as a generalization of the TDC (see Schmidt and Stadtmüller (2006)). It can be estimated with non-parametric distribution models. These models have desirable statistical properties that are inherited to the TDC and it is used to construct extreme value functions. To estimate tail copulae and the TDCs, results of the Extreme Value Theory (EVT) have been used. A number of results on copulae have been extended to tail copulae and TDCs, (see Nelsen (1999), Chapter (2) and Schmidt and Stadtmüller (2006), theorems (1-3)). In the following proposition, we use the results found in these references. Given that all four tail copulae were defined as the exact limit of copula functions and considering that the converse part of Sklar's theorem states that any copulae can be either disributional, survival, LU or UL copula, we have that all tail copulae share the same properties.

1.2.1. ***Properties of Tail Copula and the Tail Dependence Coefficient.*** Now we present some important properties of Tail Copula Λ, these properties apply to any tail copula Λ and in particular to the 4 tail copulae defined before. (see Schmidt and Stadtmüller (2006), theorems (1-3)) .

Proposition 6. *Let Λ be any tail copula according to definition (5) and $(u, v)' \in \overline{\mathbb{R}}_+^2$, then Λ has the following properties:*

*i)(**Fréchet Bounds**)* $0 \leq \Lambda(u, v) \leq \min\{u, v\}$.

*ii) (**Groundedness**)* $\Lambda(u, 0) = \Lambda(0, v) = 0$ *and for* $(u, v)' \in \mathbb{R}_+^2$*:* $\Lambda(u, \infty) = u$ *and* $\Lambda(\infty, v) = v$.

*iii) (**Monotonicity**) Λ is non-decreasing and Lipschitz continuous.*

iv) For $a,\ b > 0$: $\min\{a, b\}\Lambda(u, v) \leq \Lambda(au, bv) \leq \max\{a, b\}\Lambda(u, v)$*. This property implies*

*a) (**Homogeneity**)* $\Lambda(tu, tv) = t\Lambda(u, v)$.

b) $\min\{u,v\}\lambda \leq \Lambda(u,v) \leq \max\{u,v\}\lambda$, *hence tail independence implies* $\Lambda(u,v) = 0$ *for* $(u,v) \in [0,\infty)^2$.

c) $\Lambda(u,u) = u\lambda$.

d) $\Lambda(u,v) > 0$ *for all* u, $v \neq 0$ *or* $\Lambda(u,v) = 0$ *for all* u, v. *If for some* $(u_0,v_0) \in \mathbb{R}_+^2 := (o,\infty)^2$ *we have* $\Lambda(u_0,v_0) > 0$ *then*

for u, $v \neq 0 : \Lambda(u,v) > 0$.

v) ***(Uniformity)*** *For* $(u,v)' \in \mathbb{R}_+^2$ *the convergence of* Λ *in definition (5) is locally uniform in* $\mathbb{R}_+^2$.

vi) ***(2-increasing and strict monotonicity)*** *For* $(\overline{u},\overline{v})' \in \overline{\mathbb{R}}_+^2$ *such that* $u \leq \overline{u}$ *and* $v \leq \overline{v} : \Lambda(\overline{u},\overline{v}) - \Lambda(\overline{u},v) - \Lambda(u,\overline{v}) + \Lambda(u,v) \geq 0$, *further to this* Λ *is strictly monotonic, that is if* $u < \overline{u}$ *and* $v < \overline{v}$ *then* $\Lambda(u,v) < \Lambda(\overline{u},\overline{v})$.

vii) In definition (5) it is suficient that the limit exists on the unitary circle $(u^2 + v^2 = 1)$ *to ensure its existence on* $\overline{\mathbb{R}}_+^2$.

viii) $\frac{\partial}{\partial u}\Lambda(u,v)$ *exists for almost all* u *on* $\mathbb{R}_+$ *and* $\frac{\partial}{\partial v}\Lambda(u,v)$ *for almost all* v *on* $\mathbb{R}_+$ *and they both lie on* $[0,1]$. *In addition, both derivatives taken as univariate functions of the other variable (not the one with respect to whom we differentiate) are defined and non-decreasing almost everywhere.*

Further to this proposition, it has been proved that the tail copula Λ exists if the corresponding F lies in the domain of attraction of some max-estable extreme value distribution (see Resnick (1987)).

2. **Modelling Tail Dependence**

We now present the tail copulae associated to some of the most important copulae models used in finance and in other mathematical related areas. To obtain these results we use propositions (5) and (6). The tail dependence coefficients correspond to the case $u = v = 1$.

2.0.2. *Fundamental copula.* The fundamental copulae are three copula which represent three extreme cases of dependence between random variables, the independence, the perfect positive dependence and the perfect negative dependence cases.

If X and Y are independent the copula associated to them, any of the four types of copula, is $C(u,v) = u \cdot v$ which is known as the independence copula. The corresponding tail copulae are all equal to zero. As expected in this case all four TDCs are zero.

If $X = Y$ or they have a perfect positive dependence, their corresponding copula is $C(u,v) = \min\{u,v\}$ known as the comonotonicity copula, the corresponding tail copulae are:

$$\begin{aligned}\Lambda_L(u,v) &= \Lambda_U(u,v) = \min\{u,v\},\\ \Lambda_{LU}(u,v) &= \Lambda_{UL}(u,v) = 0.\end{aligned}$$

If $X = -Y$ or they have a perfect negative dependence, their corresponding copula is $C(u,v) = \max\{u+v-1,0\}$, which is known as the countermonotonicity copula with corresponding tail copulae:

$$\begin{aligned}\Lambda_L(u,v) &= \Lambda_U(u,v) = 0,\\ \Lambda_{LU}(u,v) &= \Lambda_{UL}(u,v) = \min\{u,v\}.\end{aligned}$$

2.0.3. *Implicit Copulae.* Implicit copulae are implied from bivariate distribution functions through the converse statement of Sklar's theorem, (see Sklar (1959)). In the two examples we consider no closed form of the copula can be obtained but the corresponding copula can be expressed in terms of integrals of the bivariate densities. The examples are the Gaussian copula and the Student's t with ν degrees of freedom, in both copulae we assume a correlation of $0 < \rho < 1$, and the distributional copulae can be expressed in terms of their corresponding densities

$$C_\rho(u,v) = \int_{-\infty}^{G^{-1}(u)} \int_{-\infty}^{G^{-1}(v)} f_\rho(s,t)\,ds\,dt.$$

All Gaussian tail copula $\Lambda_{\rho}^{Ga}(u,v)$ is equal to zero and the Student's t tail copulae are

$$\begin{aligned}
\Lambda_{L,\rho}^{t}(u,v) &= \Lambda_{U,\rho}^{t}(u,v) \\
&= u \cdot t_{\nu+1}\left(-\sqrt{\frac{\nu+1}{1-\rho^2}}\left(\left(\frac{u}{v}\right)^{\frac{1}{\nu}}-\rho\right)\right) \\
&\quad + v \cdot t_{\nu+1}\left(-\sqrt{\frac{\nu+1}{1-\rho^2}}\left(\left(\frac{v}{u}\right)^{\frac{1}{\nu}}-\rho\right)\right), \\
\Lambda_{LU,\rho}^{t}(u,v) &= \Lambda_{UL,\rho}^{t}(u,v) = \Lambda_{L,-\rho}^{t}(u,v).
\end{aligned}$$

with $t_{\nu+1}$ the standard Student's t distribution function with $\nu+1$ degrees of freedom.

2.0.4. *Explicit Copulae.* Explicit copulae are copulae that have simple closed forms The examples of explicit copulae we consider are the Gumbel, Clayton, Frank, the Generalized Clayton and the Marshall-Olkin copulae (see Schmidt and Stadtmüller (2006)).

Gumbel Copula. For $\theta \geq 1$, the Gumbel Copula has the form:

$$C_{\theta}^{Gu}(u,v) = \exp[-((-\ln u)^{\theta} + (-\ln v)^{\theta})^{\frac{1}{\theta}}].$$

The corresponding upper tail copula is $\Lambda_{U,\theta}^{Gu}(u,v) = u + v - (u^{\theta} + v^{\theta})^{\frac{1}{\theta}}$, the other tail copulae are equal to zero.

Clayton Copula. For $\theta \in [-1,\infty)\backslash\{0\}$ the Clayton Copula is:

$$C_{\theta}^{Cl}(u,v) = [\max\{u^{-\theta} + v^{-\theta} - 1, 0\}]^{-\frac{1}{\theta}}.$$

The Clayton lower tail copula is $\Lambda_{L,\theta}^{Cl}(u,v) = (u^{-\theta} + v^{-\theta})^{-\frac{1}{\theta}}$, the other tail copulae are zero.

Frank Copula. For $\theta \in \mathbb{R}$ the Frank Copula is defined as:

$$C_{\theta}^{Fr}(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{(\exp(-\theta u)-1)(\exp(-\theta v)-1)}{(\exp(-\theta)-1)}\right).$$

The four corresponding tail copulae are equal to zero.

Generalized Clayton Copula. For $\theta > 0$ and $\delta \geq 1$ the Generalized Clayton Copula is given by:

$$C^{GC}_{\theta,\delta}(u,v) = \{[(u^{-\theta}-1)^{\delta} + (v^{-\theta}-1)^{\delta}]^{\frac{1}{\delta}} + 1\}^{-\frac{1}{\theta}}.$$

The corresponding tail copulae are:

$$\begin{aligned} \Lambda^{GC}_{L,\theta,\delta}(u,v) &= (u^{-\theta\delta} + v^{-\theta\delta})^{-\frac{1}{\theta\delta}}, \\ \Lambda^{GC}_{U,\theta,\delta}(u,v) &= u + v - (u^{\delta} + v^{\delta})^{\frac{1}{\delta}}, \\ \Lambda^{GC}_{LU,\theta,\delta} &= \Lambda^{GC}_{UL,\theta,\delta} = 0. \end{aligned}$$

Marshall-Olkin Copulae. The survival Marshall-Olkin Copulae is

$$\overline{C}^{MO}(u,v) = \min\{uv^{1-\alpha_2}, u^{1-\alpha_1}v\}.$$

The corresponding lower tail copula is $\Lambda^{MO}_{L}(u,v) = \min\{\alpha_1 u, \alpha_2 v\}$, the other tail copulae are equal to zero.

With the exception of the Student's t case all LU and UL are equal to zero. Hence if we want to model data with LU or UL tail dependence we should not use any of these copulae as distributional copula, instead they can be used LU or UL copula, with the help of Table (1) we can do this without obtaining the corresponding copulae again.

3. **Conclusions**

In the present paper we have developed theory to measure the whole tail dependence structure between random variables using copula, tail copula and the tail dependence coefficients. We have developed theory to encompass the negative dependence case which has been overseen in copula theory.

In the first section we obtained several theoretical results regarding these concepts. The most important one is the Sklar theorem for LU and UL probability functions which we stated and proved. This theorem complements the Sklar's theorem for distribution and survival functions and guarantees the existence of LU and UL copulae CLU and CUL.

Together with the distributional (©) and the survival copulae, these two types characterise the dependence structure between random variables. With them it is possible to analyse negative dependence including negative tail dependence. We carried out a thorough analysis of the relationships among the four types of copulae associated to a random vector.

We obtained several results regarding copula and introduced the concept of transverse copula which comes from the relationship found for copula. This concept characterises the dependence structure between copulae.

After this we defined the four corresponding types of tail copula and the tail dependence coefficients for which we studied the main properties. We obtain some equivalent expressions for tail copula that are useful to obtain tail copula associated to copula models.

In the second section we used the results obtained in the first section to analyse copula models. We obtained expressions for the four types of copula, tail copula and the tail dependence coefficients. The examples we analysed correspond to three types of copula models. The fundamental copula, which encompass the perfect dependence cases. The implicit copulae, which arise when a particular distribution function is assumed, we analysed the Gaussian and the Student'-((t)) copula. Finally we studied explicit copulae for which closed form expressions exist, we studied four Archimedean and the Marshall Olkin copula.

We found that the Studen-((t)) copula model is the only one that accounts for all the types of tail dependence where as the Independence, the Gaussian and the Frank copula models do not account for any type of tail dependence. Further to this only the countermonotonicity and the Student-((t)) copula models account for negative tail dependence. The values of the LU and UL tail copula are equal to zero for all the other copula models. This means that when these models are used there is an underlying assumption of no negative tail dependence that might bear unwanted side effects.

References

Bollerslev (1990) Modelling the Coherence in Short-run Nominal Exchange Rates: A Multivariate Generalized ARCH Model *Review of Economics and Statistics,* **72** (3), 498-505.

Carr, P. and Madan, D. (2001) Towards a Theory of Volatility Trading *Handbooks in Mathematical Finance: Option Pricing, Interest Rates and Risk Management,* 458-476.

Coles, S., Heffernan, J., Tawn, J. (1999) Dependence Measures for Extreme Value Analyses *Extremes* **2**, 339-365

Demeterfi, K., Derman, E., Kamal, M. and Zou, J. (1999) A Guide to Volatility and Variance Swaps *Journal of Derivatives,* **6** (4), 9-32

Dobric, J. and Schmid, F. (2005) Nonparametric Estimation of the Lower Tail Dependence Coefficient l in Bivariate Copulas. *Journal of Applied Statistics* **32(4)**, 387-407

Dominici, D. (2003) The Inverse of the Cumulative Standard Normal Probability Function *Integral Transforms and Special Functions*, **14**, Issue 4, 281-292.

Einmahl, J., de Haan, L. and Li, D. (2006) Weighted Approximations of Tail Copula Processes with Application to Testing the Multivariate Extreme Value Condition. *Annals of Statistics* **34** No. 4, 1987-2014

Embrechts, P., Klüppelberg, C., Mikosch, T. (1997). Modelling Extremal Events for Insurance and Finance, Springer, Berlin

Embrechts, P., Lindskog, F., McNeil, A. (2003) Modelling Dependencies with Copulas and Applications to Risk Management. *Handbook of Heavy Tailed Distributions in Finance*, ed S. Rachev, Elsevier, Chapter 8, 329-384.

Embrechts, P., McNeil, A. and Straumann, D. (2002) Correlation and Dependence in risk management: properties and pitfalls, in Dempster, M.A.H. (ed.), *Risk Management: Value at Risk and Beyond* (Cambridge University Press, Cambridge) 176-223

Fils-Villetard, A., Guillou, A. and Segers, J. (2008) Projection Estimators of Pickands Dependence Functions. *The Canadian Journal of Statistics* **36**, No. 3, 369-382

Fischer, M. and Dörflinger, M. (2006) A Note on a Non-Parametric Tail Dependence Estimator *Discussion Paper* **76**.

Frahm, G., Junker, M. and Scmidt, R. (2005) Estimating the Tail-Dependence Coefficient: Properties and Pitfalls. *Insurance: Mathematics and Economics* **37**, 80-100.

Fortin, I., Kuzmics, C., (2002) Tail-Dependence in Stock-Return Pairs *Institute for Advanced Studies, Economic Series 126.*

Genest, C., Ghoudi, K. and Rivest, L.-P.(1995) A semiparametric estimation procedure of dependence parameters in multivariate familiees of distributions. *Biometrika* **82**, 543-552.

Genest, C. and Segers, J. (2009) Rank-Based Inference for Bivariate Extreme-Value copulas. *The Annals of Statistics*, **37.**

Härdle, W., Okhrin, O., Okhrin, Y.(2009) Modeling Dependencies in Finance using Copulae. *Applied Quantitative Finance (2nd. Edition) W. Härdle, N. Hautsch, L. Overbeck (eds),* Springer Verlag.

Huang X. (1992) Statistics of bivariate extreme values. PhD Thesis, *Tinbergen Institute Research Series* **Volume 22** Thesis Publishers and Tinbergen Institute, Rotterdam.

Joe, H., Smith, R.L., Weissman, I. (1992) Bivariate threshold models for extremes. *Journal of the Royal Statistical Society,* Series B 54, 171-183.

Karmakar, S. and Simonovic, S.P. (2009) Bivariate Flood Frequency Analysis. Part2: A Copula-Based Approach with Mixed Marginal Distributions *Journal of Flood Risk Management* **2**, Issue 1, 32-44.

McNeil, A.J. and Frey, R. (2000) Estimation of tail-related risk measures for heteroscedastic financial time series: An extreme value approach, *Journal of Empirical Finance* **7**, 271-300.

McNeil, A., Frey, R., Embrechts, P. (2005) Quantitative Risk Management: Concepts, Techniques and Tools, Princeton Series in Finance

Nelsen, R. B. (1999) An introduction to Copulas, Springer

Ng, W. L. and Peng, Y. (2009) Analysing Asymmetric Market Dependence with Volatility Indices and Mixed Copula *Working Paper Series, WP0-09, CCFEA, University of Essex, UK.*

Patton, A.J. On the Out-of-Sample-Importance of Skewness and Asymmetric Dependence for Asset Allocation *Journal of Financial Econometrics* **2**, No. 1, 130-168 (not used yet)

Pickands, J. (1981) Multivariate extreme value distribution. *Proceedings of the 43rd session ISI (Buenos Aires)* 859-878.

Resnick, S. (1987) Extreme values, regular variation, and point processes. Springer Verlag, New York, NY.

Rigobon, R. (2003) On the Measurement of the International Propagation of Shocks: is the Transmission Stable?, *Journal of International Economics,* **61**, 261-283

Roncalli, T., Riboulet, G., Nikeghbali, A., Durrleman, V., Bouyé, E. (2001) Copulas: an Open Field for Risk Management, *Warwick Business School, Financial Econometrics Research Centre, Working Papers.* wp01-01

Schmidt, T. (2007) Coping with Copulas *Published as Chapter 1 in Copulas: From Theory to Application in Finance (2007), Edited by Rank, J.* d-fine GmbH.

Schmidt, R. & Stadtmüller, U. (2006) Non-Parametric Estimation of Tail Dependence, *Scand. J. Statist.* **33**, 307-335.

Schoepe, W.(2003) Extremes of Random Processes in Condensed Matter Physics, *Bilbao*

Schweizer, B. and Sklar, A. (1983) Probabilistic Metric Spaces. New York: North Holland

Segers, J. (2007) Non-Parametric Inference for Bivariate Extreme-Value Copulas, *CentER Discussion Paper 2004-91, Tilburg University,* www.center.nl.

Shaw, W. T., and Lee, K.T.A. (2007) Copula Method vs Canonical Multivariate Distributions: The Multivariate Student T Distribution with General Degrees of Freedom. *Working Paper*

Shorak, G.R. and Weller, J.A. (1986) Empirical Processes with Applications to Statistics, Wiley.

Sklar, A. (1959) Fonctions de répartition à n dimensions et leurs margès *Publications de l'Institute de Statistique de L'Université de Paris* **8**, 229-231

Steinbrecher, G. and Shaw, W. (2008) Quantile Mechanics, *European Journal of Applied Mathematics*, **19**, No. 2, 87-112.

Zhang, M.-H. (2007) Modelling total tail dependence along diagonals, *Insurance: Mathematics and Economics,* **42**, 77-80

Capéraà, P., Fougères, A.-L. (2000) Estimation of a Bivariate Extreme Value Distribution. *Extremes* **3** 311-329

Capéraà, P., Fougères, A.-L. and Genest, C. (1997) A Nonparametric Estimation Procedure for Bivariate Extreme Value Copulas *Biometrika* **84** No.4 (Sep., 1997) 567-577

Capéraà, P., Fougères, A.-L. and Genest, C. (2000) Bivariate Distributions with Given Extreme Value Attractor *Journal of Multivariate Analysis,* **72**, 30-49

de Barra, G.(1981) Measure Theory and Integration. Ellis Horwood

Einmahl, J., de Haan, L. and Piterbarg, A. (2001) Nonparametric Estimation of the Spectral Measure of an Extreme Value Distribution, *The Annals of Statistics,* **29**, No. 5, 1401-1423

Einmahl, J., de Haan, L. and Sinha, A. (1997) Estimating the Spectral Measure of an Extreme Value Distribution, *Stochastic Processes and their Applications,* **70,** 143-171

Jackweerth, J.C. (1999) Option-Implied Risk-Neutral Distributions and Implied Binomial Trees: A Literature Review *The Journal of Derivatives* **7**, 66-82

Harrison, J. M. and Pliska S. R. (1981) Martingales and Stochastic Integrals in the Theory of Continuous Trading *Stochastic Porcesses and their Applications* **11** (3) 215-260

Klüppelberg, C. Kuhn, G. and Peng, L. (2006) Estimating the Tail Dependence Function of an Elliptical Distribution, *Bernoulli.* **13** No. 1, 229-251.

Klüppelberg, C., Kuhn, G. and Peng, L. (2008) Semi-Parametric Models for the Multivariate Tail Dependence Function - the Asymptotically Dependent Case, *Scand. J. Statist.* **35** No. 4, 701-718

The CBOE Volatility Index "VIX White paper" (2009) available at: http://www.cboe.com/micro/vix/vixwhite.pdf

Wheeden, R. & Zygmund, A. (1977). Measure and Integral- an introduction to real analysis. A series of monographs and textbooks: pure and applied mathematics. Marcel Dekker Inc., New York, NY.

S&P 500 "Fact Sheet" (2009) available at: http://www2.standardandpoors.com/spf/pdf/index/SP_500_Factsheet.pdf

Markose, S. and Alentorn, A. (2005) The Generalized Extreme Value Distribution (GEVD), Implied Tail Index and Option Pricing, *University of Essex, Department of Economics Discussion Paper, Number 594*

Alentorn, A.. and Markose, S (2007) Generalized Extreme Value Distribution and Extreme Economic Value at Risk (EE-VaR) *University of Essex, Department of Economics Discussion Paper*

Reiss, R.-D. and Thomas, M. (2007) Statistical Analysis of Extreme Values: with Applications to Insurance, Finance, Hydrology and other fields, Springer

University of Essex, UK

E-mail address: ysalaz@essex.ac.uk